Ordinary generating functions

Useful identities to recall (valid for small x):

$$\frac{1-x^k}{1-x} = \sum_{n=0}^{k-1} x^n \qquad (1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{-k}{n}(-x)^n = \sum_{n=0}^{\infty} \binom{k+n-1}{n} x^n$$

Study sequences of numbers $h_0, h_1, h_2, \ldots$.
For example, h_i may count # of ways to express i as a sum on 7 non-negative integers.

Idea: Use h_i as coefficients of a polynomial

$$g(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_tx^t + \cdots$$

Use notation

$$h_i = [x^i]g(x).$$

The main idea is that it may be sometimes easier to calculate h_i for all i at once rather than for each i separately.

1: Find the generating function it a closed form for the sequence

$$1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$$

Solution:

$$g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}.$$

2: Find the generating function in a closed form for the sequence

$$1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \cdots$$

Solution:

$$g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

It can be summed only for $|x| < 1$ but we typically ignore for what x it exists. Say we always take sufficiently small $x \neq 0$ where the sum converges.

3: Find the generating function in a closed form for the sequence

$$1\ 3\ 3\ 1\ 0\ 0\ 0\ 0 \cdots$$

Solution:

$$g(x) = 1 + 3x + 3x^2 + x^3 = \binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3 = (1+x)^3.$$

Closed form of generating function is particularly handy when multiplying or adding generating functions. Lets also try the opposite - i.e. get a coefficient from a closed form of a generating function.

4: Find coefficient of x^{2020} of $g(x)$, that is $[x^{2020}]g(x)$, where
(a) $g(x) = (1-2x)^{5000}$
(b) $g(x) = \frac{1}{1+3x}$
(c) $g(x) = \frac{1}{(1+5x)^2}$

Solution: (a)

$$g(x) = (1-2x)^{5000} = \sum_k \binom{5000}{k}(-2x)^k$$

Hence $[x^{2020}]g(x) = \binom{5000}{2020}(-2)^{2020}$
(b)

$$g(x) = \frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum_i (-3x)^i$$

Hence $[x^{2020}]g(x) = (-3)^{2020}$
(c)

$$\frac{1}{1+5x} = \frac{1}{1-(-5x)} = 1 - 5x + (-5x)^2 + (-5x)^3 + \cdots$$

Then

$$\frac{1}{1+5x} \cdot \frac{1}{1+5x} = (1 - 5x + (-5x)^2 + \cdots) \cdot (1 - 5x + (-5x)^2 + \cdots)$$

Coefficient x^{2020} is then $2021 \cdot (-5)^{2020} = 2021 \cdot 5^{2020}$ since there are 2021 choices for picking the ways of combine x^{2020}.
Alternatively using Newton's Binomial Theorem:

$$\frac{1}{(1+5x)^2} = (1+5x)^{-2} = \sum_i \binom{-2}{i}(-5x)^i = \sum_i (-1)^i \binom{2+i-1}{i}(5x)^i$$

This give $[x^{2020}]g(x) = \binom{2021}{2020}5^{2020}$.

5: Let $k \in \mathbb{N}$ be fixed. Let h_t be the number of integer solutions of

$$e_1 + e_2 + e_3 + \cdots + e_k = t,$$

where $e_1 \geq 0, e_2 \geq 0, \ldots, e_k \geq 0$. Find a closed form for the generating function.

Solution: First we compute $h_t = \binom{t+k-1}{k-1}$. This makes generating function

$$g(x) = \sum_{t=0}^{\infty} \binom{t+k-1}{k-1} x^t = \left(\frac{1}{1-x}\right)^k$$

Notice that

$$g(x) = \left(\frac{1}{1-x}\right)^k = (1 + x + x^2 + \cdots) \cdot (1 + x + x^2 + \cdots) \cdots (1 + x + x^2 + \cdots).$$

How does it compute the number of solutions as coefficient of x^t? The exponents correspond to the solutions! That is

$$x^t = x^{e_1} \cdot x^{e_2} \cdot x^{e_3} \cdot \cdots \cdot x^{e_k}.$$

This corresponds exactly to solutions of

$$e_1 + e_2 + e_3 + \cdots + e_k = t,$$

6: Find a (more) closed form for the following generating function and try to find interpretation as solutions

$$(1 + x + x^2 + x^3 + x^4 + x^5) \cdot (x + x^2) \cdot (1 + x + x^2 + x^3 + x^4)$$

Solution:

$$g(x) = \frac{1 - x^6}{1 - x} \cdot \frac{x(1 - x^2)}{1 - x} \cdot \frac{1 - x^5}{1 - x}$$

We compute $x^t = x^{e_1} x^{e_2} x^{e_3}$. This makes $t = e_1 + e_2 + e_3$ subject to $0 \leq e_1 \leq 5$, $1 \leq e_2 \leq 4$, $0 \leq e_3 \leq 4$.

7: Write down the generating series for counting the number of possibilities to pay t cents using 1, 5 and 25 cent coins.

Solution:

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{25}}$$

Notice we are solving equations

$$t = e_1 + 5e_2 + 25e_3$$

where $0 \leq e_1, e_2, e_3$.

8: Count the number of ways to make a pack of n fruits if

- # of apples is even
- # of bananas is a multiple of 5
- # at most 4 oranges
- # 0 or 1 pear

Write as generating function $g(x)$ and read $[x^n]g(x)$.

Solution:

$$\begin{aligned} g(x) &= (1+x^2+x^4+\cdots)\cdot(1+x^5+x^{10}+\cdots)\cdot(1+x+x^2+x^3+x^4)\cdot(1+x) \\ &= \frac{1}{1-x^2}\cdot\frac{1}{1-x^5}\cdot\frac{1-x^5}{1-x}\cdot(1+x) \\ &= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}\binom{n+1}{n}x^n = \sum_{n=0}^{\infty}(n+1)x^n \end{aligned}$$

Hence $[x^n]g(x) = n+1$, which is the solution.

Notice the generating function works something like

(or or or) and (or or or) and (or or or)

9: 20 students, how many ways to pick 7 who get A? Solve this using generating functions as well as without it. Build the generating function be deciding for every student individually if the student is getting A or not.

Solution:

$$(1+x)(1+x)\cdots(1+x) = (1+x)^{20}$$

Coefficient is then easily $\binom{20}{7}$.

10: 20 students, how many ways to distribute 50 identical candies to the students? Use generating functions.

Solution:

$$g(x) = (1+x+x^2+\cdots)^{20} = \left(\frac{1}{1-x}\right)^{20} = (1-x)^{-20} = \sum_{i=0}\binom{20+i-1}{i}(-1)^{20}(-x)^{20}$$

Then

$$[x^{50}]g(x) = \binom{20+50-1}{20}$$

Notice we solved a slightly more general problem. If we want, we can try to solve it only for 50 candies. Then

$$g(x) = (1+x+x^2+\cdots+x^{50})^{20} = \left(\frac{1-x^{51}}{1-x}\right)^{20} = (1-x^{51})^{20}(1-x)^{-20} = (1-x^{51})^{20}\sum_{i=0}\binom{20+i-}{i}$$

If we are interested only in $[x^{50}]g(x)$, the terms $(1-x^{51})$ each contribute just 1 and the coefficient is $\binom{69}{20}$.

11: Find a generating function that counts how many ways is it possible to score 6 points in basketball. In basketball, a throw can give 1,2, or 3 points. A 'way' to get 6 points is to say how many throws of each score happen.

Solution: Enumerate solution:

$$1+1+1+1+1+1 \qquad 1+1+1+1+2 \qquad 1+1+2+2 \qquad 2+2+2$$
$$1+1+1+3 \qquad 3+3 \qquad 1+2+3$$

Important is number of 1, 2 and 3 points. Contribution of each of them to the result is:

$$1:0,1,2,3,4,5,6 \qquad 2:0,2,4,6 \qquad 3:0,3,6$$

Write the following generating function, where we use the contributions in the exponent:

$$(\underbrace{1+x+x^2+x^3+x^4+x^5+x^6}_{\text{contribution of 1}})\cdot(\underbrace{1+x^2+x^4+x^6}_{\text{contribution of 2}})\cdot(\underbrace{1+x^3+x^6}_{\text{contribution of 3}})$$

Paste to WolframAlpha and obtain:

$x^{18}+x^{17}+2x^{16}+3x^{15}+4x^{14}+5x^{13}+7x^{12}+7x^{11}+8x^{10}+8x^9+8x^8+7x^7+7x^6+5x^5+4x^4+3x^3+2x^2+x$

How is x^6 obtained? We need to pick x^{a_1} from the contribution of 1, then x^{a_2} from the contribution of 1, and x^{a_3} from the contribution of 3. Moreover, we need $a_1+a_2+a_3=6$. This is exactly solving the previous question. In particular, we are getting

$$x^{6+0+0}+x^{4+2+0}+x^{2+4+0}+x^{0+6+0}+x^{3+0+3}+x^{0+0+6}+x^{1+2+3}=7x^6.$$

Is the answer good also for x^5 or x^7?

12: Find generating function, where h_n counts the number of ways to score n points in basketball.

Solution:

$$g(x)=(1+x+x^2+x^3+\cdots)\cdot(1+x^2+x^4+x^6+\cdots)\cdot(1+x^3+x^6+x^9+\cdots)=\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}$$

13: Compute generating function counting number of ways of getting sum h_i on two dices.

Solution:

$$(x+x^2+x^3+x^4+x^5+x^6)^2$$

Notice that

$$(x+x^2+x^3+x^4+x^5+x^6)^2=(1+x)^2\cdot(x+x^3+x^5)^2.$$

This means that rolling two dice is the same as two coin flips and two rolls of a special dice that has values 1,3, and 5.

14: Determine the generating series for partitions. Partitions is the number of ways to write n as a sum on at most n non-negative integers that decrease in size. That is,

$$n = x_1 + x_2 + x_3 + \cdots + x_n,$$

where $x_1 \geq x_2 \geq x_3 \geq \cdots \geq 0$. Let h_n be the number of partitions on n, write the generating function for sequence h_n.
Hint: In partition is important how many times is each number used. Solution is a big product.

Solution: We create functions counting how many times is each digit used

$$\begin{aligned}
1 &: 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \\
2 &: 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2} \\
3 &: 1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}
\end{aligned}$$

In total, we obtain

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

15: Give generating function for the following sequence

$$1, 2, 3, 4, 5, 6, \ldots$$

Hint: Derivative

Solution: Use a simple function and since the polynomial are equal, you can use derivative

$$\begin{aligned}
\frac{1}{1-x} &= x^0 + x^1 + x^2 + x^3 + \cdots + x^n + \cdots \\
\frac{1}{(1-x)^2} &= x^0 + 2x^1 + 3x^2 + \cdots + nx^{n-1} + \cdots
\end{aligned}$$

So the generating function is $g(x) = \frac{1}{(1-x)^2}$.

16: Determine the generating function for the number h_n of integral solutions of

$$2e_1 + 11e_2 + e_3 + 7e_4 = n,$$

where $0 \le e_1$, $2 \le e_2$, $0 \le e_3 \le 10$ and $1 \le e_4 \le 5$.
Use it to compute h_{31}.
Do not try to evaluate all h_n, just get the generating function and get h_{31}. But get a closed form function, i.e. no infinite sums or infinite products.

Solution: We can rewrite the question using $a_1 = 2e_1$, $a_2 = 11e_2$ and $a_4 = 7e_4$ as

$$a_1 + a_2 + e_3 + a_4 = n,$$

where a_1 is a multiple of 2, a_2 is a multiple of 11 and a_4 is a multiple of 7. So we get

$$(1+x^2+x^4+\cdots)(x^{22}+x^{33}+x^{44}+\cdots)(1+x+x^2+x^3+\cdots+x^{10})(x^7+x^{14}+x^{21}+\cdots+x^{35}).$$

This simplifies as

$$(1+x^2+x^4+\cdots)x^{22}(1+x^{11}+x^{22}+\cdots)(1+x+x^2+x^3+\cdots+x^{10})x^7(1+x^7+x^{14}+\cdots+x^{28}).$$

Hence

$$\begin{aligned} g(x) &= x^{29} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^{11}} \cdot \frac{1-x^{11}}{1-x} \cdot \frac{1-x^{35}}{1-x^7} \\ &= x^{29} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{1-x^{35}}{1-x^7} \end{aligned}$$

Now we can get

$$\begin{aligned} [x^{31}]g(x) &= [x^2]\left(\frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{1-x^{35}}{1-x^7}\right) \\ &= [x^2]\left(\frac{1}{1-x^2} \cdot \frac{1}{1-x}\right) = [x^2]((1+x^2+\cdots)\cdot(1+x+x^2+\cdots)) = 2. \end{aligned}$$

Solution using generating functions Idea: Find generating function $g(x)$ for h_n and then read $[x^n]g(x)$.

Recall

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} \binom{-n}{k}(-rx)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k$$

17: Solve the following recurrence using generating functions

$$h_n = 5h_{n-1} - 6h_{n-2}$$

$h_0 = 1$ and $h_1 = -2$.

Solution: Observe that

$$h_n - 5h_{n-1} + 6h_{n-2} = 0$$

We write h_n into a generating function and try to use the previous observation.

$$\begin{array}{rcccccccccc} g(x) & = & h_0 & + & h_1 x & + & h_2 x^2 & + & h_3 x^3 & + & \cdots \\ -5xg(x) & = & & - & 5h_0 x & - & 5h_1 x^2 & - & 5h_2 x^3 & - & \cdots \\ 6x^2 g(x) & = & & & & & 6h_0 x^2 & + & 6h_1 x^3 & + & \cdots \end{array}$$

By summing all three equations we get

$$(1-5x+6x^2)g(x) = h_0 + (h_1 - 5h_0)x + (h_2 - 5h_1 + 6h_0)x^2 + (h_3 - 5h_2 + 6h_1)x^3 + \cdots$$
$$(1-5x+6x^2)g(x) = h_0 + (h_1 - 5h_0)x$$
$$(1-5x+6x^2)g(x) = 1 - 7x$$
$$g(x) = \frac{1-7x}{1-5x+6x^2}$$

Now we will use partial fractions. Notice $6x^2 - 5x + 1 = (1-2x)\cdot(1-3x)$. We want

$$g(x) = \frac{1-7x}{1-5x+6x^2} = \frac{c_1}{1-2x} + \frac{c_2}{1-3x}$$

The solution for c_1 and c_2 is obtained from

$$1 - 7x = c_1(1-3x) + c_2(1-2x) = c_1 + c_2 - c_1 3x - c_2 2x = (c_1 + c_2) - (3c_1 + 2c_2)x$$

and we need to solve system of 2 equations:

$$1 = c_1 + c_2 \qquad\qquad -7 = 3c_1 + 2c_2$$

Solution is $c_1 = 5$ and $c_2 = -4$. Hence

$$g(x) = \frac{5}{1-2x} - \frac{4}{1-3x}$$
$$= 5\sum_{i=0}^{\infty}(2x)^i - 4\sum_{i=0}^{\infty}(3x)^i$$

Hence

$$h_n = 5\cdot 2^n - 4\cdot 3^n.$$